of the unit square, and is such that $\alpha(C)$ is well defined for any bivariate copula. Examples are measures of monotone association such as Spearman’s $\rho_S$ and the correlation of normal scores $\rho_N$. The theory below can be adapted to other types of functionals.

Assume that $\alpha$ is twice continuously differentiable in the region where it is non-zero, and that $a_{12} = \frac{\partial^2 \alpha}{\partial u \partial v}$ is integrable on $(0,1)^2$. With $(u_0, v_0)$ and $(u, v)$ in the interior of the unit square (or on the boundary if $a(\cdot)$ is finite everywhere),

$$a(u, v) - a(u_0, v) - a(u, v_0) + a(u_0, v_0) = \int_{u_0}^{u} \int_{v_0}^{v} a_{12}(s, t) \, dsdt.$$

After substituting for $a(u, v)$ and changing the order of integration, (5.15) is the same as:

$$\alpha^*(C) = \int_0^1 \int_0^1 a_{12}(s, t) C(s, t) \, dsdt + \int_{u_0}^{1} \int_{v_0}^{1} a_{12}(s, t) \, dsdt$$

$$- \int_{u_0}^{1} \int_{v_0}^{1} t a_{12}(s, t) \, dsdt - \int_{u_0}^{1} \int_{v_0}^{1} s a_{12}(s, t) \, dsdt$$

$$+ \int_{0}^{1} a(u_0, v) \, dv + \int_{0}^{1} a(u, v_0) \, du - a(u_0, v_0).$$

(5.16)

The sample version of (5.15) is

$$\hat{\alpha} = \alpha(\hat{C}_n) = \int_0^1 \int_0^1 a(u, v) \, d\hat{C}_n(u, v) = n^{-1} \sum_{i=1}^{n} a\left(\frac{r_{i1} - \frac{1}{2}}{n}, \frac{r_{i2} - \frac{1}{2}}{n}\right)$$

$$= \alpha^*(\hat{C}_n) + O_p(n^{-1})$$

$$= \int_{0}^{1} \int_{0}^{1} a(u, v) \, d\hat{C}_n(u, v) + O_p(n^{-1}).$$

Note that with $S_n = [(r_{i1} - 1)/n, r_{i1}/n] \times [(r_{i2} - 1)/n, r_{i2}/n],

$$\int_{(r_{i1} - 1)/n}^{r_{i1}/n} \int_{(r_{i2} - 1)/n}^{r_{i2}/n} a(u, v) \, d\hat{C}_n(u, v) = n^{-1} a\left(\frac{r_{i1} - \frac{1}{2}}{n}, \frac{r_{i2} - \frac{1}{2}}{n}\right),$$

and from a Taylor expansion to first order,

$$\left| \int_{S_n} a(u, v) \, \hat{C}_n(u, v) \, dv - n^{-1} a\left(\frac{r_{i1} - \frac{1}{2}}{n}, \frac{r_{i2} - \frac{1}{2}}{n}\right) \right| \leq n^{-2} \max_{(u, v) \in S_n} \left\{ \frac{\partial a}{\partial u} \frac{\partial a}{\partial v} \right\}.$$

For asymptotics, with the continuous mapping theorem,

$$n^{1/2}[\alpha(\hat{C}_n) - \alpha(C)] = \int_{[0,1]^2} a_{12}(u, v) n^{1/2}[\hat{C}_n(u, v) - C(u, v)] \, dv + O_p(n^{-1/2})$$

$$- \int_{[0,1]^2} a_{12}(u, v) \, g_C(u, v) \, dv.$$

Hence $\alpha(\hat{C}_n)$ is asymptotically normal. If (5.16) doesn’t hold, asymptotic normality holds under some conditions and the asymptotic representation has a different form.

An example of a tail-weighted measure of dependence is $\text{Cor}[b(U), b(V)|U > \frac{1}{2}, V > \frac{1}{2}]$ if $b$ is increasing on $[\frac{1}{2}, 1]$ with $b(\frac{1}{2}) = 0$. For the upper semi-correlation of normal scores, $b(u) = \Phi^{-1}(u)$ for $\frac{1}{2} \leq u < 1$ (but the second version $\alpha^*$ after integration by parts might not be valid). This conditional correlation involves $E[b^{m_1}(U)b^{m_2}(V)|U > \frac{1}{2}, V > \frac{1}{2}]$, with $(m_1, m_2) \in \{(1, 1), (2, 0), (0, 2), (1, 0), (0, 1)\}$, that is, (5.15) for five different $a(\cdot)$ functions.